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2003 J. Phys. A: Math. Gen. 36 7533

(http://iopscience.iop.org/0305-4470/36/27/306)

INSTITUTE OF PHYSICS PUBLISHING JOURNAL OF PHYSICS A: MATHEMATICAL AND GENERAL

J. Phys. A: Math. Gen. **36** (2003) 7533–7543 PII: S0305-4470(03)58887-4

Universality of stretched Gaussian asymptotic behaviour for the fractional Fokker–Planck equation in external force fields

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Received 28 January 2003, in final form 28 April 2003
Published 25 June 2003
Online at stacks.iop.org/JPhysA/36/7533

Abstract
We introduce a heterogeneous fractional Fokker–Planck equation (HFFPE) on heterogeneous fractal structure media describing systems involving external force fields. The HFFPE is shown to obey the generalized Einstein relation, and its stationary solution is the Boltzmann distribution. It is proven that the asymptotic shape of its solution is a stretched Gaussian and that its solution can be expressed in the form of a function of a dimensionless similarity variable for constant and generic potentials with polar singularity at origin.

PACS numbers: 05.10.Gg, 05.45−a

1. Introduction

In recent years, much attention has been paid to physical systems driven by transport mechanisms other than ordinary diffusion, in particular, diffusion in a disordered crystalline medium. This leads to many anomalous physical properties [1]. In the physics of complex systems, anomalous transport properties and their description have attracted considerable interest. Applications have been found in a wide field ranging from physics and chemistry to biology and medicine [1–5]. Anomalous diffusion in one dimension is characterized by the occurrence of a mean square displacement of the form

$$X^2 = \langle X^2\rangle(t) = \langle\langle \Delta x\rangle^2\rangle(t) = \frac{2K_\gamma}{\Gamma(1+\gamma)}t^\gamma \tag{1.1}$$

which deviates from the linear Brownian dependence on time [3]. In equation (1.1), the anomalous diffusion coefficient K_γ is introduced, which has the dimension $[K_\gamma] = \mathrm{cm}^2\,\mathrm{s}^{-\gamma}$.

0305-4470/03/277533+11$30.00

Recently in the literature, in order to describe diffusion processes in disordered media, some authors have proposed an extension of the Fokker–Planck (FP) equation, which is called the fractional Fokker–Planck equation (FFPE) [6–13].

A FFPE describing anomalous transport close to thermal equilibrium has been presented recently [6]. Since it describes subdiffusion in the force-free case, it involves a strong, i.e. slowly decaying, memory. In their original framework [9], Metzler *et al* gave a seminal framework and investigated the anomalous diffusion and relaxation involving external fields with the one-dimensional FFPE for one variable

$$\dot{W}(x,t) = {}_0D_t^{1-\gamma} L_{\mathrm{FP}} W \tag{1.2}$$

with respect to its physical properties. Here, $W(x,t)$ is the probability density function (pdf) at position x at time t, and the FP operator

$$L_{\mathrm{FP}} = \frac{\partial}{\partial x}\left(\frac{V'(x)}{m\eta_\gamma} + K_\gamma \frac{\partial}{\partial x}\right) \tag{1.3}$$

with the external potential $V(x)$ [15], contains the anomalous diffusion constant K_γ and the anomalous friction coefficient η_γ with the dimension $[\eta_\gamma] = \mathrm{s}^{\gamma-2}$. Herein, m denotes the mass of the diffusion particle, and

$${}_0D_t^{1-\gamma} W = \frac{1}{\Gamma(\gamma)}\frac{\partial}{\partial t}\int_0^t \mathrm{d}\tau \frac{W(x,\tau)}{(t-\tau)^{1-\gamma}}. \tag{1.4}$$

The interesting part has asymptotic behaviour $\log W(x,t) \sim -c\xi^u$ where $\xi \equiv x/t^\alpha/2 \gg 1$ which is expected to be universal. Here, $u = 1/(1-\alpha/2)$ with the anomalous diffusion exponent α which is the order of the fractional derivative [14].

In [30], the FFPE

$${}_0D_t^\alpha W(x,t) = Gx^{-\theta'} L_{\mathrm{FP}} W(x,t) \tag{1.5}$$

has been presented, where $G > 0$ is to be determined, $\theta' \geqslant 0$ is a parameter, $0 < \alpha < 1$, if $\theta' = 0$ then $\alpha = \gamma$ and equation (1.5) reduces to

$${}_0D_t^\alpha W(x,t) = GL_{\mathrm{FP}} W(x,t) \tag{1.6}$$

which leads to the FFPE (1.2). It is proven that for the FFPE (1.6), its solution has asymptotic behaviour

$$\log W(x,t) \sim -c\xi^u \tag{1.7}$$

where

$$\xi \equiv x/t^{\gamma/2} \gg 1 \qquad u = 1\Big/\left(1-\frac{\gamma}{2}\right) \tag{1.8}$$

and possesses a scaling variable for constant potential, linear potentials, logarithm potentials and harmonic potentials.

El-Wakil and Zahranit [12] have discussed the fact that anomalous diffusion in a heterogeneous fractal medium in one dimension is characterized by the occurrence of a mean square displacement of the form

$$X_\theta^2 = \langle\langle \Delta x\rangle^2\rangle \sim x^{-\theta} t^\gamma \qquad 0 < \gamma \leqslant 1 \quad \theta = d_w - 2. \tag{1.9}$$

We call θ a heterogeneous exponent and γ is known to be the diffusion exponent.

By a simple scaling consideration as in [17], according to equation (1.9) we require that $x^{-\theta}t^\gamma \sim x^2$, i.e. $x \sim t^{\frac{\gamma}{(2+\theta)}}$. Thus, we have that $X_\theta^2 \sim t^{\frac{2}{(2+\theta)}\gamma}$. Hence we can rewrite equation (1.9) as

$$X_\theta^2 = \langle\langle \Delta x\rangle^2\rangle = \frac{2K_\gamma^\theta}{\Gamma(1+\gamma_\theta)} t^{\gamma_\theta} \tag{1.10}$$

where $\gamma_\theta = \frac{2\gamma}{2+\theta}$. In equation (1.10) the anomalous diffusion coefficient K_γ^θ is introduced, which has the dimension $\left[K_\gamma^\theta\right] = \mathrm{cm}^2\ \mathrm{s}^{-2\gamma/(2+\theta)}$. In equation (1.9), $d_w > 2$ is the anomalous diffusion exponent [1].

It is easy to see that for $\theta \to 0$, equation (1.10) reduces to equation (1.1) and K_γ^θ to K_γ.

So we naturally ask the following question. Does the FFPE with multi-parameters $\alpha, \theta, \theta', \gamma$ and μ

$$ {}_0D_t^\alpha W(x,t) = Gx^{-\theta'} L_{\mathrm{FP}}^\mu W(x,t) \qquad \mu > 0 \tag{1.11}$$

still possess the stretched Gaussian and a scaling variable for constant and generic potentials where $L_{\mathrm{FP}}^\mu = \frac{\partial}{\partial x}\left(\frac{V'(x)}{m\eta_\gamma} + K_\gamma^\theta \frac{\partial}{\partial x} x^{-\mu}\right)$?

The main purpose of this paper is to solve this problem. By using the heuristic argument of Giona and Roman [17], we introduce a FFPE on heterogeneous fractal structures which can lead to the FFPE (1.5). It is proven that the asymptotic shape of its solution is a stretched Gaussian and that its solution can be expressed in the form of a function of a dimensionless similarity variable for constant and generic potentials.

2. The derivation of the FFPE with external potentials

In this section, we derive the fractional diffusion equation involving the external potential $V(x)$ by using the heuristic argument of Giona and Roman [17].

The relationship between the total flux of probability current $S(x,t)$ from time $t = 0$ to time t and the average probability density $W(x,t)$, considered as the input and the output of the fractal system [18] (cf [17]), should satisfy the following equation:

$$ \int_0^t S(x,\tau)\,\mathrm{d}\tau = x^{d_f-1} \int_0^t K(t,\tau) W(x,\tau)\,\mathrm{d}\tau \tag{2.1}$$

where d_f is the fractal dimension of the system considered. This is a conservation equation containing an explicit reference to the history of the diffusion process on a fractal structure. Since we are dealing with stationary processes, we expect $K(t,\tau)$ to be a function of difference $t-\tau$ only, i.e. $K(t,\tau) = K(t-\tau)$. $K(t,\tau)$ is the diffusion kernel. We assume that diffusion sets are underlying fractals (underlying fractals denote self-similar sets in [19] or net fractals in [20, 21]) and the diffusion kernel on the underlying fractal should behave as

$$ K(t-\tau) = \frac{A_\alpha}{(t-\tau)^\alpha} \tag{2.2}$$

with $0 < \alpha < 1$, where α is a diffusion exponent and A_α is a constant that can be determined [21].

On the other hand, on the above fractional structure we propose that the probability current $S(x,t)$ satisfies the following structure equation:

$$ S(x,t) = Bx^{d_f-1}x^{-\theta'} \frac{\partial}{\partial x}\left(\frac{V'(x)}{m\eta_\gamma} + K_\gamma^\theta \frac{\partial}{\partial x} x^{-\mu}\right) W(x,t) \qquad \mu > 0 \tag{2.3}$$

i.e.

$$ S(x,t) = Bx^{d_f-1}x^{-\theta'} L_{\mathrm{FP}}^\mu W(x,t)$$

where $B > 0$ is to be determined, θ' is still a parameter and $L_{\mathrm{FP}}^\mu = \frac{\partial}{\partial x}\left(\frac{V'(x)}{m\eta_\gamma} + K_\gamma^\theta \frac{\partial}{\partial x} x^{-\mu}\right)$ is the FP operator.

From equations (2.1), (2.2) and (2.3), we have

$$ {}_0D_t^\alpha W(x,t) = Gx^{-\theta'} L_{\mathrm{FP}}^\mu W(x,t) \tag{2.4}$$

where

$$G = B/\Gamma(1-\alpha)A_\alpha > 0. \tag{2.5}$$

We call equation (2.4) a heterogeneous fractional Fokker–Planck equation (HFFPE).

It is easy to see that for $\mu \to 0$, L_{FP}^{μ} reduces to the L_{FP} and equation (2.4) to FFPE (1.5).

Furthermore, we suppose that the pdf $W(x,t)$ satisfies the following normalization conditions:

$$\int_0^\infty \mathrm{d}x \cdot x^{d_f-1} W(x,t) = 1 \tag{2.6}$$

and the extraction of moments $\langle(\Delta X)^n\rangle$ is defined by

$$\langle(\Delta X)^n\rangle = \int_0^\infty \mathrm{d}x \cdot x^{d_f-1} x^n W(x,t). \tag{2.7}$$

Remark 1. We know that the integral equation

$$\frac{1}{\Gamma(\nu)}\int_0^t \frac{g(\tau)\,\mathrm{d}\tau}{(t-\tau)^{1-\nu}} = G(t) \qquad t > 0 \tag{2.8}$$

where $0 < \nu < 1$, is called Abel's equation and for any summable function $g(t)$ it has a unique solution [25]

$$g(t) = \frac{1}{\Gamma(1-\nu)}\frac{\mathrm{d}}{\mathrm{d}t}\int_0^t \frac{G(\tau)\,\mathrm{d}\tau}{(t-\tau)^{\nu}}. \tag{2.9}$$

Conversely, if equation (2.9) holds, then equation (2.8) is satisfied for $G(t)\epsilon I_0^\alpha(L_1)$, i.e. there exists a summable function $h(t)$ such that

$$G(t) = \frac{1}{\Gamma(\nu)}\int_0^t \frac{h(\tau)\,\mathrm{d}\tau}{(t-\tau)^{1-\nu}} \qquad t > 0. \tag{2.10}$$

Thus, from equation (2.4) we have

$$W(x,t) = \frac{1}{\Gamma(\alpha)}\int_0^t Gx^{-\theta'}\frac{L_{\mathrm{FP}}^{\mu}W(x,\tau)}{(t-\tau)^{1-\alpha}}\,\mathrm{d}\tau \tag{2.11}$$

for $W(x,t)\epsilon I_0^\alpha(L_1)$. Hence

$$\dot{W}(x,t) = \frac{1}{\Gamma(\alpha)}\frac{\partial}{\partial t}\int_0^t Gx^{-\theta'}\frac{L_{\mathrm{FP}}^{\mu}W(x,\tau)}{(t-\tau)^{1-\alpha}}\,\mathrm{d}\tau$$

i.e.

$$\dot{W}(x,t) = Gx^{-\theta'}{}_0D_t^{1-\alpha}L_{\mathrm{FP}}^{\mu}W(x,t) \tag{2.12}$$

where

$${}_0D_t^{1-\alpha}W = \frac{1}{\Gamma(\alpha)}\frac{\partial}{\partial t}\int_0^t \frac{W(x,\tau)\,\mathrm{d}\tau}{(t-\tau)^{1-\alpha}}. \tag{2.13}$$

Especially, when $G = 1, \theta' = 0$ and $\alpha = \gamma$

$$\dot{W}(x,t) = {}_0D_t^{1-\alpha}L_{\mathrm{FP}}^{\mu}W. \tag{2.14}$$

Equation (2.14) is just the one-dimensional FFPE (1.2). The derivation of equation (2.14) is different from those in [9, 26]. This shows that the asymptotic equation (2.3) is reasonable. Conversely, if $W(x,0) = 0$ and equation (2.12) holds, then equation (2.4) is satisfied if $Gx^{-\theta'}L_{\mathrm{FP}}^{\mu}W(x,t)$ is a summable function.

If in the presence of an external nonlinear and time-independent field the stationary state is reached, then S must be constant. Thus, if $S = 0$ for any x, it vanishes for all x [15], and the stationary solution is given by $L_{FP}^{\mu} W(x,t) = 0$, i.e. $V'(x)W_{st}/(m\eta_\gamma) + K_\gamma^\theta\, \mathrm{d}(x^{-\mu}W_{st})/\mathrm{d}x = C$ (constant) from which the exponential result

$$W_{st}(x) = A^\delta x^\mu \left\{ C' \exp\left\{ -\left(\int x^\mu V'(x)\,\mathrm{d}x \Big/ \left(m\eta_\gamma K_\gamma^\theta\right)\right)\right\} + C\left[\frac{m\eta_\gamma}{A^\delta x^\mu V'(x)} - \mathrm{e}^{-\int x^\mu V'(x)\,\mathrm{d}x/(m\eta_\gamma K_\gamma^\theta)} \int \mathrm{e}^{\int x^\mu V'(x)\,\mathrm{d}x/(m\eta_\gamma K_\gamma^\theta)}\, \mathrm{d}\left(\frac{m\eta_\gamma}{A^\delta x^\mu V'(x)}\right)\right]\right\} \qquad (2.15)$$

can be inferred, where $\delta = \mu/(\mu+1) < 1$ and C' is a constant. Requiring, in analogy with the standard case, that W_{st} is given by the generalized Boltzmann distribution in a heterogeneous fractal medium, i.e.

$$W_{st} \propto x^\mu \exp\left\{ -\int x^\mu V'(x)\,\mathrm{d}x/(k_B T)\right\} \qquad (2.16)$$

C must be equal to zero. Thus, the generalized Einstein–Stocks–Smoluchowski relation

$$K_\gamma^\theta = \frac{k_B T}{m\eta_\gamma} \qquad (2.17)$$

is readily recovered. Thus, the FFPE (2.4) obeys some generalized fluctuation–dissipation theorem.

It is easy to see that for $\mu \to 0$ the generalized Boltzmann distribution (2.16) and Einstein relation (2.17) reduce to the well-known Boltzmann distribution $W_{st} \propto \exp\{-V(x)/(k_B T)\}$ and Einstein relation $K_\gamma = k_B T/(m\eta_\gamma)$, respectively.

We can anticipate the relation between exponents $\alpha, \gamma, \theta, \mu$ and θ' by simple scaling considerations. From equation (2.4) we see that $t^\alpha \sim x^{2+\theta'+\mu}$, and according to equation (1.9) we require that $t^\gamma \sim x^{2+\theta}$. Thus

$$\alpha = \gamma\left(\frac{2+\mu+\theta'}{2+\theta}\right). \qquad (2.18)$$

Corresponding to the value $\theta' = 0$, the new lower bound $\alpha \geqslant \gamma$ follows from equation (2.18). From the upper bound $\alpha < 1$, equation (2.18) follows the upper bound for θ'

$$\theta' < \frac{(2+\theta)}{\gamma} - (\alpha - \mu). \qquad (2.19)$$

3. Solutions and the properties of the HFFPE

In this section we give the solutions of the FFPE (2.4).

It can be shown that the Laplace transform of the fractional derivative ${}_0D_t^\alpha W(x,t)$ is

$$\mathcal{L}\left[{}_0D_t^\alpha W(x,t)\right] = s^\alpha W(x,s).$$

Using this result, from equation (2.4) we obtain

$$s^\alpha W(x,s) = Gx^{-\theta'} \frac{\partial}{\partial x}\left[\frac{V'(x)}{m\eta_\gamma} W(x,s) + K_\gamma^\theta \frac{\partial x^{-\mu} W(x,s)}{\partial x}\right] \qquad (3.1)$$

where $W(x,s)$ denotes the Laplace transform of $W(x,t)$.

Case I: Constant potential. First we consider the constant potential $V(x) =$ constant, leading to the force-free case. In this case, equation (3.1) reduces to

$$x^2 \frac{\partial^2 W(x,s)}{\partial x^2} - 2\mu x \frac{\partial W(x,s)}{\partial x} - [q x^{2+\mu+\theta'} - \mu(\mu+1)]W(x,s) = 0 \qquad (3.2)$$

where $q = s^{\alpha}/G_{\gamma},\ G_{\gamma} = GK_{\gamma}^{\theta}$. In order to solve equation (3.2), it is convenient to perform the transformation

$$y = A(s)x^{v} \qquad W(x,s) = y^{\delta}Z(y) \tag{3.3}$$

to cast equation (3.2) into the second-order Bessel equation as

$$y^2\frac{\mathrm{d}^2 Z}{\mathrm{d}y^2} + y\frac{\mathrm{d}Z}{\mathrm{d}y} - (\lambda^2 + y^2)Z = 0 \tag{3.4}$$

with parameter λ^2 under the following conditions:

$$v = \frac{2+\mu+\theta'}{2} \qquad A(s) = \frac{q^{1/2}}{v} \qquad \delta = \frac{1}{2v}(1+2\mu) \tag{3.5}$$

where

$$\lambda^2 = \frac{2\delta\mu}{v} - \delta\left(\delta - \frac{1}{v}\right) - \frac{\mu(\mu+1)}{v^2} = 1/4v^2. \tag{3.6}$$

The solution of this equation, satisfying the summability condition $\lim_{x\to+\infty} W(x,t) = 0$, i.e. $W(x,s) = 0\,(x \to +\infty)$, is given by $Z(y) = C(s)K_{\lambda}(y)$ since $y \to +\infty$

$$K_{\lambda}(y) = \mathrm{e}^{-y}\left(\frac{\pi}{2y}\right)^{1/2}\left[1 + O\left(\frac{1}{y}\right)\right] \tag{3.7}$$

in the domain $|\arg y| < \frac{3}{2}\pi$, where K_{λ} is the modified Bessel function of second order and $C(s)$ is to be determined. So the solution of equation (3.2) is given by

$$W(x,s) = C(s)y^{\delta}K_{\lambda}(y) \qquad y = A(s)x^{v} \tag{3.8}$$

where

$$C(s) = G's^{(\alpha d_f/2v)-1} \qquad G' = v^{1-d_f/v}\big/\big[C_{\lambda}\big(GK_{\gamma}^{\theta}\big)^{d_f/2v}\big] \tag{3.9}$$

ensures the normalization condition (2.6) of $W(x,t)$, i.e. $\int_0^{\infty}\mathrm{d}x \cdot x^{d_f-1}W(x,s) = 1/s$, if and only if $0 < \lambda < 1$. $C_{\lambda} = \int_0^{\infty}\mathrm{d}y \cdot y^{\frac{d_f}{v}+\delta-1}K_{\lambda}(y)$ is a constant since [28]

$$\int_0^{\infty}\mathrm{d}y \cdot y^{\nu}K_{\lambda}(ay) = 2^{\nu-1}a^{-\nu-1}\Gamma\left(\frac{1+\nu+\lambda}{2}\right)\Gamma\left(\frac{1+\nu-\lambda}{2}\right).$$

It is easy to see from equations (1.10) and (2.7) (when $n = 2$) that

$$\alpha = \left(\frac{2+\mu+\theta'}{2+\theta}\right)\gamma \qquad B = \frac{A_{\alpha}\Gamma(1-\alpha)}{v^2}\left(2\frac{C_{\lambda}}{C'_{\lambda}}\right)^{v}K_{\gamma}^{\theta\,v-1} \tag{3.10}$$

since $\int_0^{\infty}\mathrm{d}x \cdot x^{d_f-1}x^2W(x,s) = 2K_{\gamma}^{\theta}\big/s^{2\gamma/(2+\theta)+1}$ corresponding to the Laplace transform of $\langle\langle\Delta x\rangle^2\rangle(t)$. $C'_{\lambda} = \int_0^{\infty}\mathrm{d}y \cdot y^{\frac{d_f+2}{v}+\delta-1}K_{\lambda}(y)$.

Let us now discuss the asymptotic behaviour of $W(x,t)$ as predicted by equation (3.8). We have

$$W(x,s) \approx G''s^{-(1-\alpha d_f/2v)}\frac{1}{(xs^{\alpha/2v})^{\kappa}}\exp\{-(xs^{\alpha/2v}/G''')^{v}\} \tag{3.11}$$

for $|xs^{\alpha/2v}| \gg 1$, where $G'' = \sqrt{\frac{\pi}{2}}v^{1-(d_f+\kappa)/v}/C_{\lambda}\big(GK_{\gamma}^{\theta}\big)^{(d_f+\kappa)/2v} > 0$, $G''' = \big[v\sqrt{GK_{\gamma}^{\theta}}\big]^{1/v}$, $\kappa = v\big(\frac{1}{2} - \delta\big)$.

Using the same method as in [17], we expect that

$$W(x,t) \sim t^{-(\alpha d_f/2v)}(x/X_{\theta})^{\delta'}\exp\{-\mathrm{const} \times (x/X_{\theta})^{u'}\} \tag{3.12}$$

when $x/X_\theta \sim x/t^{\gamma/(2+\theta)} \gg 1$ and $t \to +\infty$. The Laplace transform of equation (3.12) can be evaluated by applying the method of steepest descent and the result compared with equation (3.11). This yields

$$u' = v \Big/ \left(1 - \frac{\alpha}{2}\right) \tag{3.13}$$

and

$$\delta' = u' \left[\tfrac{1}{2}(\alpha d_f/v - 1) - \kappa\right]. \tag{3.14}$$

It is interesting that if $\mu = \theta = \theta' = 0$, we have $u' = 1\big/\left(1 - \frac{\gamma}{2}\right)$ and $\delta' = u'\left[\frac{1}{2}(\gamma d_f - 1)\right]$, which are the same of those of FFPE [30].

Using the inversion theorem of Laplace transform, from equation (3.11) we have (see the appendix)

$$W(x, t) \approx \frac{G''}{\pi} t^{-\alpha d_f/2v} z^{-\kappa} f(z) \tag{3.15}$$

where

$$f(z) = \sum_{n=0}^{\infty} C_n(\beta) z^{nv} \qquad z = x/t^{\alpha/2v} \qquad \beta = \alpha(d_f - \kappa)/2v - 1.$$

In particular, for $z \ll 1$, $W(x, t) \approx \frac{G'' C_0(\beta)}{\pi} t^{-\alpha d_f/2v} z^{-\kappa}$; thus, if $\kappa > 0$, $W(x, t)$ diverges on the origin.

Case II: Generic potentials. Now we discuss the general case. In this case, equation (3.1) becomes

$$x^2 \frac{\partial^2 W(x, s)}{\partial x^2} + \left(\frac{V'(x) x^{1+\mu}}{m_\gamma^\theta} - 2\mu\right) x \frac{\partial W(x, s)}{\partial x} - \left[q x^{2+\mu+\theta'} - \mu(\mu + 1) - \frac{V''(x) x^{2+\mu}}{m_\gamma^\theta}\right] W(x, s) = 0 \tag{3.16}$$

where $q = s^\alpha \big/ G K_\gamma^\theta$, $m_\gamma^\theta = m\eta_\gamma K_\gamma^\theta$.

We assume that the generic external potential at the origin is given by

$$V(x) = b_p x^{-p} \quad (p \neq 0) \qquad \text{and} \qquad V(x) = b_p ln(x) \quad (p = 0) \tag{3.17}$$

where $b_p \neq 0$.

In order to solve equation (3.16), it is convenient to perform the transform $y = A(s)x^v$, $W(x, s) = y^\delta Z(y)$ to cast equation (3.16) into the second-order Bessel equation as

$$y^2 \frac{d^2 Z}{dy^2} + y \frac{dZ}{dy} - \left(\lambda_g^2 + y^2\right) Z = 0 \tag{3.18}$$

with parameter λ_g^2 under the following conditions:

$$v = \frac{2 + \mu + \theta'}{2} \qquad A = q^{1/2}/v = s^{\alpha/2}/v\sqrt{G K_\gamma^\theta} \tag{3.19}$$

$$\delta_g = \frac{1 + 2\mu}{v} - \frac{1}{2v m_\gamma^\theta} V'\left(\left(\frac{y}{A}\right)^{\frac{1}{v}}\right) \left(\frac{y}{A}\right)^{\frac{1+\mu}{v}} \tag{3.20}$$

where

$$\lambda_g^2 = \frac{2\mu\delta_g}{v} - \frac{\mu(\mu+1)}{v^2} - \delta_g\left(\delta_g - \frac{1}{v}\right) - \frac{V''\left(\left(\frac{y}{A}\right)^{\frac{1}{v}}\right)\left(\frac{y}{A}\right)^{\frac{2+\mu}{v}}}{v^2 m_\gamma^\theta}$$
$$- \frac{\delta_g}{v m_\gamma^\theta} V'\left(\left(\frac{y}{A}\right)^{\frac{1}{v}}\right)\left(\frac{y}{A}\right)^{\frac{1+\mu}{v}}. \tag{3.21}$$

Noting for $\mathrm{Re}(s^{\alpha/2})/y \gg 1$, $y/|A| \ll 1$, if $\mu - p > 0$, and for $\mathrm{Re}(s^{\alpha/2})/y \ll 1$, $y/|A| \gg 1$, if $\mu - p < 0$, from equation (3.17), we have $\left(\frac{y}{A}\right)^{\frac{1+\mu}{v}} V'\left(\left(\frac{y}{A}\right)^{\frac{1}{v}}\right) \ll 1$ and $\left(\frac{y}{A}\right)^{\frac{2+\mu}{v}} V''\left(\left(\frac{y}{A}\right)^{\frac{1}{v}}\right) \ll 1$. Then from equations (3.20) and (3.21), we have

$$\delta_g \approx \frac{1+2\mu}{2v} = \delta \qquad \lambda_g \approx \frac{1}{4v^2} = \lambda. \tag{3.22}$$

Thus, equation (3.18) can be replaced by

$$y^2\frac{\mathrm{d}^2 Z}{\mathrm{d}y^2} + y\frac{\mathrm{d}Z}{\mathrm{d}y} - (\lambda^2 + y^2)Z = 0. \tag{3.4}$$

Hence, if $\mu \neq p$, we have

$$w(x,t) \sim t^{-(\alpha d_f/2v)}\left(\frac{x}{X_\theta}\right)^{\delta'} \exp\left\{-\mathrm{const} \times \left(\frac{x}{X_\theta}\right)^{u'}\right\} \tag{3.23}$$

where $x/X_\theta \sim x/t^{\gamma/(2+\theta)} \gg 1$ and $t \to +\infty$,

$$u' = v\Big/\left(1 - \frac{\alpha}{2}\right) \tag{3.24}$$

$$\delta' = u'\left[\frac{1}{2}(\alpha d_f/v - 1) - \kappa\right] \tag{3.25}$$

$$\kappa = v\left(\frac{1}{2} - \delta\right) \tag{3.26}$$

and

$$W(x,t) \approx \frac{G''}{\pi} t^{-\alpha d_f/2v} z^{-\kappa} f(z) \tag{3.27}$$

where

$$f(z) = \sum_{n=0}^{\infty} C_n(\beta) z^{nv} \qquad z = x/t^{\alpha/2v} \qquad \beta = \alpha(d_f - \kappa) - 1. \tag{3.28}$$

If $\mu = p \neq 0$, for $\mathrm{Re}(s^{\alpha/2}/y) \gg 1$, since

$$\left(\frac{y}{A}\right)^{\frac{1+\mu}{v}} V'\left(\left(\frac{y}{A}\right)^{\frac{1}{v}}\right) \approx -pb_p \qquad \left(\frac{y}{A}\right)^{\frac{2+\mu}{v}} V''\left(\left(\frac{y}{A}\right)^{\frac{1}{v}}\right) \approx p(p+1)b_p$$

we have

$$\delta_g \approx \frac{1+2\mu}{2v} + \frac{pb_p}{2v m_\gamma^\theta} = \delta_p \tag{3.29}$$

$$\lambda_g \approx \frac{1}{4v^2}\left(1 + \frac{pb_p}{m_\gamma^\theta}\right)^2 = \lambda_p^2. \tag{3.30}$$

Similarly, if $\mu = p = 0$, for $\mathrm{Re}(s^{\alpha/2}/y) \gg 1$, since

$$\left(\frac{y}{A}\right)^{\frac{1+\mu}{v}} V'\left(\left(\frac{y}{A}\right)^{\frac{1}{v}}\right) = b_0 \qquad \left(\frac{y}{A}\right)^{\frac{2+\mu}{v}} V''\left(\left(\frac{y}{A}\right)^{\frac{1}{v}}\right) = -b_0$$

we have

$$\delta_g \approx \frac{1+}{2v} - \frac{b_0}{2vm_\gamma^\theta} = \delta_0 \tag{3.31}$$

$$\lambda_g \approx \frac{1}{4v^2}\left(1 - \frac{b_0}{m_\gamma^\theta}\right)^2 = \lambda_0^2. \tag{3.32}$$

In this case, equation (3.18) can be replaced by

$$y^2\frac{d^2Z}{dy^2} + y\frac{dZ}{dy} - \left(\lambda_p^2 + y^2\right)Z = 0. \tag{3.33}$$

As in the discussion of case I, we have

$$W(x,t) \sim t^{-(\alpha d_f/2v)}(x/X_\theta)^{\delta'}\exp\{-\text{const} \times (x/X_\theta)^{u'}\} \tag{3.34}$$

where $x/X_\theta \sim x/t^{\gamma/(2+\theta)} \gg 1$ and $t \to +\infty$,

$$u' = v \bigg/ \left(1 - \frac{\alpha}{2}\right) \tag{3.35}$$

$$\delta' = u'\left[\frac{1}{2}(\alpha d_f/v - 1) - \kappa_p\right] \tag{3.36}$$

$$\kappa_p = v\left(\frac{1}{2} - \delta_p\right) \tag{3.37}$$

and

$$W(x,t) \approx \frac{G''}{\pi} t^{-\alpha d_f/2v} z^{-\kappa_p} f(z) \tag{3.38}$$

where

$$f(z) = \sum_{n=0}^{\infty} C_n(\beta) z^{nv} \qquad z = x/t^{\alpha/2v} \qquad \beta = \alpha(d_f - \kappa_p) - 1. \tag{3.39}$$

Remark 2. It is worth pointing out that, for $\theta \to 0$, since equation (1.10) reduces to equation (1.1) and $K_\gamma^\theta \to K_\gamma$, with the above discussion and results, the solution of FFPE (2.4) with respect to equation (1.1) has the asymptotic behaviour

$$W_0(x,t) \sim t^{-(\alpha d_f/2v)}(x/X_\theta)^{\delta'}\exp\{-\text{const} \times (x/X_\theta)^{u'}\} \tag{3.40}$$

when $x/X \sim x/t^{\gamma/2} \gg 1$ and $t \to +\infty$. It possesses a scaling variable, i.e.

$$W_0(x,t) \approx \frac{G_0''}{\pi} t^{-\alpha d_f/2v} z^{-\kappa_0} f_0(z) \tag{3.41}$$

$$f_0(z) = \sum_{n=0}^{\infty} C_n(\beta_0) z^{nv} \qquad z = x/t^{\alpha/2v} \qquad \beta_0 = \alpha(d_f - \kappa_0)/2v - 1$$

where

$$\alpha = v\gamma \qquad v = (2 + \mu + \theta')/2 \tag{3.42}$$

and

$$u' = v \bigg/ \left(1 - \frac{\alpha}{2}\right) \qquad \delta' = u'\left[\frac{1}{2}(d_s - 1) - \kappa_0\right]. \tag{3.43}$$

For the constant potentials and the generic potentials of equation (3.17) with $u \neq p$

$$\kappa_0 = v\left(\frac{1}{2} - \delta\right) \qquad \delta = \frac{1+2\mu}{2v} \tag{3.44}$$

and for the generic potentials of equation (3.17) with $u = p \neq 0$

$$\kappa_0 = v\left(\tfrac{1}{2} - \delta_p\right) \tag{3.45}$$

where δ_p is given by equations (3.29) or (3.31) for $\mu = p \neq 0$ and $\mu = p = 0$, respectively.

$d_s = \gamma d_f$ is called the fraction or spectral dimension [22, 23].

4. Conclusion and discussion

In order to describe anomalous diffusion processes involving external potential fields on heterogeneous fractal structures, we introduce a HFFPE. Its solution possesses the following properties. The necessary and sufficient condition of its stationary solution, which is the generalized Boltzmann distribution, is that the generalization of the Einstein relation is also preferred as the Stocks–Einstein–Smoluchowski relation holds

$$K_\gamma^\theta = k_B T / m\eta_\gamma$$

for the anomalous coefficients K_γ and η_γ. For $\theta \to 0$ and $\theta' = 0$ this result is consistent with that in [9].

There exists an intrinsic relationship $\alpha = \gamma\left(\frac{2+\mu+\theta'}{2+\theta}\right)$ between $\gamma, \alpha, \theta, \mu$ and θ'. γ, θ and α are structural parameters of the underlying fractal structure and α can be determined explicitly; see [18–20]. If $\theta' = 0$ and $\mu = \theta$ then $\alpha = \gamma$.

The solution of the FFPE has asymptotic behaviour $\log W(x,t) \sim -c\xi^u$ where $\xi \equiv x/t^{\gamma/2} \gg 1$, $u = v\big/\left(1 - \frac{\alpha}{2}\right)$, $v = (2 + \mu + \theta')/2$, and possesses a scaling variable for constant and generic potentials.

If θ tends to zero, we obtain the corresponding results for the FFPE on homogeneous fractal structures.

Acknowledgment

This work was supported by NSFC and by the Shanghai Priority Academic Discipline (Project no 10271031).

Appendix

Using the inversion theorem of Laplace transform, from equation (3.11) we have

$$\begin{aligned} W(x,t) &= \frac{G_0}{\pi}\int_0^\infty \tau^\beta \exp\{-[t\tau + G_2\tau^\alpha \cos\alpha\pi]\} \sin[G_2\tau^\alpha \sin\alpha\pi - \beta\pi]\,\mathrm{d}\tau \\ &= \frac{G_0}{\pi} t^{-\beta-1}\int_0^\infty y \exp\{-[y + G_2(y/t)^\alpha \cos\alpha\pi]\} \sin[G_2(y/t)^\alpha \sin\alpha\pi - \beta\pi]\,\mathrm{d}y \end{aligned} \tag{A.1}$$

where

$$G_2 = \frac{x^{\theta'+1}}{(\theta'+1)GK_\gamma^\theta}.$$

Using the expressions of power series of entire-functions e^z, $\cos z$ and $\sin z$ and the integral representation of the Γ function, a simple integration of equation (A.1) yields the following expression with the dimensionless similarity variable $z = x/t^{\alpha/2v}$:

$$W(x,t) = \frac{G''}{\pi} t^{-\alpha d_f/2v} z^{-\kappa} f(z) \tag{A.2}$$

where

$$f(z) = \sum_{m=0}^{\infty} C_m(\beta) z^{mv}$$

$$\begin{aligned}
&C_0 = a'\Gamma(\gamma d_f/2) > 0 \qquad C_m(\beta) = a'b_m + b'a_{m-1} \\
&a_m(\beta) = \sum_{k,n\geqslant 0, 2k+1+n=m} (-1)^{k+n} \frac{a^{2k+1}b^n}{(2k+1)!n!} C^{m+1}\Gamma\left(\beta + 1 + \frac{\alpha}{2}(2k+1+n)\right) \\
&b_m(\beta) = \sum_{k,n\geqslant 0, 2k+n=m} (-1)^{k+n} \frac{a^{2k}b^n}{(2k)!n!} C^{m}\Gamma\left(\beta + 1 + \frac{\alpha}{2}(2k+n)\right) \\
&a' = -\sin\beta\pi > 0 \qquad b' = \cos\beta\pi \qquad a = \sin\frac{\alpha}{2}\pi/G''' \qquad b = \cos\frac{\alpha}{2}\pi/G'''.
\end{aligned} \tag{A.3}$$

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